

# Rational Term Rewriting Revisited: Decidability and Confluence<sup>\*</sup>

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**Abstract.** We consider a variant of rational term rewriting as first introduced by Corradini et al., i.e., we consider rewriting of (infinite) terms with a finite number of different subterms. Motivated by computability theory, we show a number of decidability results related to the rewrite relation and prove an effective version of a confluence theorem for orthogonal systems.

## 1 Introduction

Cyclic term graph rewriting [1,3,16] can be given an operational semantics based on non-cyclic terms by considering both finite and infinite terms (or trees) and defining a notion of rewriting for these. Two forms of so-called infinitary rewriting exist: (1) Corradini considers finite reductions where in each step infinitely many redexes may be rewritten simultaneously [4]. (2) Kennaway et al. allow for reductions of any countable ordinal length, but in each step only one redex may be rewritten [13,12].

The operational interpretation of cyclic term graph rewriting now takes the form of a so-called adequacy result. For Corradini's notion of infinitary rewriting, this result occurs in [5]. With regard to the notion of infinitary rewriting defined by Kennaway et al., the result can be found in [11].

Infinitary rewriting is strictly more general than any finite form of rewriting. There are, e.g., continuum many infinite terms over any signature consisting of at least two unary function symbols. Unfortunately, this means that infinitary rewriting only exists as a mathematical abstraction, not as something we can actually compute with.

To rectify the above situation, the usual approach is to consider infinite terms that can be finitely represented. In [14] this is done with respect to the work of Kennaway et al. using Turing machines and taking as a starting point methods from computable analysis [18]. In the current work, we consider finite representations within the setting of Corradini's work. More precisely, we consider

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rewriting of terms that are rational, i.e., rewriting of terms which have only finitely many different subterms. It is well-known that rational terms can be represented finitely [7].

As unraveling a cyclic term graph into an infinite term always yields a term that is rational [11], one may wonder whether we are not simply considering term graph rewriting in disguised form. However, our notion of rewriting is strictly more powerful. From the perspective of term graph rewriting, our rewrite steps combine two operations: The first operation is to transform the considered term graph into *any* other term graph, where the only restriction is that the new graph has the same unraveling. The second operation is the actual rewrite step. The first operation is atypical of graph rewriting, where transformations are usually limited to ones that reduce sharing, e.g., by means of a copying operation [1], or ones that only introduce vertical sharing [15]. Hence, it is not clear whether our notion of rewriting is still decidable.

**Related Work.** The notion of rational rewriting already occurs in two places in the literature. In [10], rewriting of so-called  $\mu$ -terms (i.e., term graphs with only vertical sharing) is dubbed “rational rewriting”. This work shares with ours the two-phase approach to rewriting as outlined above. However, a redex may only be rewritten when no back edges point to nodes on the path from the root of the graph to the redex.

Similar to us, [6,5] define rational rewriting as a restriction to rational terms of Corradini’s notion of infinitary rewriting [4]. However, decidability issues related to the rewrite relation are not considered. Furthermore, although [5] mentions a confluence result for orthogonal systems, no actual proof is provided and, hence, it is unclear to us whether an effective version of confluence is intended.

**Outline.** We both extend and restrict the notion of rational rewriting from [6,5]. We extend it in the sense that we allow for rewriting in systems with arbitrary rule sets, not just orthogonal ones, as in [6,5]. We restrict the notion of rewriting in the sense that only one rewrite rule may be employed in each rewrite step instead of arbitrary combinations of rules, as again in [6,5]. Currently, this restriction is necessary, as we do not know whether our decidability results hold without it. We hypothesize, however, that this restriction may be lifted.

The remainder of the paper is divided into two parts. In the first, we show decidability of several properties related to rational rewrite steps: Given a regular set  $\Delta$  of positions, a rewrite rule  $l \rightarrow r$ , and rational terms  $s$  and  $t$ , we show that  $s \rightarrow_{l \rightarrow r}^{\Delta} t$  is decidable, where  $\rightarrow_{l \rightarrow r}^{\Delta}$  denotes the simultaneous rewrite of the  $l \rightarrow r$ -redexes at positions  $\Delta$ . More generally, given a rational term  $s$ , a rewrite system  $\mathcal{R}$ , and regular set  $\Delta$ , we show that the set of all terms  $t$  with  $s \rightarrow_{\mathcal{R}}^{\Delta} t$  can be constructed effectively. Finally, we show that it is decidable whether a rational  $s$  is in normal form with respect to a set of rewrite rules  $\mathcal{R}$ .

In the second part of the paper, we prove for orthogonal systems that, given reductions  $s \rightarrow^* t_1$  and  $s \rightarrow^* t_2$ , we can effectively construct a rational term  $u$  and reductions  $t_1 \rightarrow^* u$  and  $t_2 \rightarrow^* u$ , i.e., an algorithm exists constructing

$u$  and the two reductions. We also show that this result cannot be extended to so-called weakly orthogonal systems.

Remark that, as we are interested in computability results, we will be explicit regarding the finite representation we use for rational terms (instead of leaving this implicit). We will work with so-called regular systems of equations (see Definition 3.1), which is one of many such finite representations [7]. Note that the same representation has been used in term graph rewriting [1].

## 2 Preliminaries

We assume basic familiarity with term rewriting [2,17]. However, to fix notation, we recall some basic definitions.

Let  $\Sigma$  be a set of function symbols and  $\mathcal{V}$  be a countable, infinite set of variables. Each function symbol  $f \in \Sigma$  is equipped with a natural number called the *arity* of  $f$ . The set of function symbols of arity  $n$  is denoted by  $\Sigma_n$ .

Let  $\mathbb{N}^*$  be the set of finite strings over the positive numbers, with  $\epsilon$  the empty string,  $\cdot$  the operator for concatenating strings, and  $<$  the prefix order on strings. We can define the set of (finite and infinite) terms as follows [9]:

**Definition 2.1.** *The set of (finite and infinite) terms  $\mathcal{T}(\Sigma, \mathcal{V})$  is the set of partial functions  $t : \mathbb{N}^* \rightarrow \Sigma \cup \mathcal{V}$  such that  $t(\epsilon)$  is defined and for all  $p \in \mathbb{N}^*$ : (1) if  $t(p)$  is defined, then  $t(q)$  is defined for all  $q < p$ ; (2) if  $t(p) \in \Sigma_n$ , then  $t(p \cdot i)$  is defined for all  $1 \leq i \leq n$  and undefined otherwise; (3) if  $t(p) \in \mathcal{V}$ , then  $t(q)$  is undefined for all  $q > p$ .*

*The domain over which  $t$  is defined is denoted by  $\text{Pos}(t)$  and the elements of the domain are called positions. A term  $t$  is finite if  $\text{Pos}(t)$  is finite; the set of finite terms is denoted by  $\mathcal{T}_{\text{fin}}(\Sigma, \mathcal{V})$ . Moreover, define  $\text{Pos}_{\Sigma}(t) = \{p \in \text{Pos}(t) \mid t(p) \in \Sigma\}$  and  $\text{Pos}_{\mathcal{V}}(t) = \{p \in \text{Pos}(t) \mid t(p) \in \mathcal{V}\}$ . Finally, let  $\mathcal{V}(t)$  denote the set of variables in  $t$ .*

For any  $f \in \Sigma_n$  and terms  $t_1, \dots, t_n \in \mathcal{T}(\Sigma, \mathcal{V})$ , the term  $t = f(t_1, \dots, t_n)$  is defined by  $t(\epsilon) = f$  and  $t(i \cdot p) = t_i(p)$  for all  $1 \leq i \leq n$  and  $p \in \mathbb{N}^*$ . A function  $\sigma : \mathcal{V} \rightarrow \mathcal{T}(\Sigma, \mathcal{V})$  is called a *substitution* and is extended to a function  $\sigma : \mathcal{T}(\Sigma, \mathcal{V}) \rightarrow \mathcal{T}(\Sigma, \mathcal{V})$  as usual.

Let the *subterm at position  $p \in \text{Pos}(t)$  in  $t \in \mathcal{T}(\Sigma, \mathcal{V})$* , denoted  $t|_p$ , be defined by  $t|_p(q) = t(p \cdot q)$ . We can now define the following:

**Definition 2.2.** *The set of regular trees or rational terms  $\mathcal{T}_{\text{reg}}(\Sigma, \mathcal{V}) \subseteq \mathcal{T}(\Sigma, \mathcal{V})$  is the set of terms with a finite number of different subterms, i.e., for each  $t$  the set  $\{t|_p \mid p \in \text{Pos}(t)\}$  is finite.*

Note that for any rational term  $t$ , we have that the set of positions  $\text{Pos}(t)$  is regular (i.e., there exists a finite automaton that recognizes  $\text{Pos}(t)$  as a language). Moreover, every finite term is regular. Finally, note that matching is decidable for rational terms [7].

### 3 Representing Rational Terms

Write  $\Sigma_{\perp}$  for the set of function symbols  $\Sigma$  extended with a fresh constant  $\perp$ . Recall that rational terms are solutions of regular systems of equations and vice versa [7]:

**Definition 3.1.** *A regular system of equations is a finite set of equations  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  with mutually distinct  $x_1, \dots, x_n \in \mathcal{V}$  and  $t_i \in \mathcal{T}_{fin}(\Sigma, \mathcal{V})$  for all  $1 \leq i \leq n$ . In  $E$ , a variable  $x_i$  is called (non-)looping if there (do not) exist  $1 \leq i_1, \dots, i_k \leq n$  with  $x_i = t_{i_1}$  and  $t_{i_j} = x_{i_{(j \bmod k)+1}}$  for all  $1 \leq j \leq k$ .*

*The domain of  $E$ , denoted  $\text{Dom}(E)$ , is the set of variables  $\{x_1, \dots, x_n\}$ ; the range of  $E$ , denoted  $\text{Ran}(E)$ , is the set of terms  $\{t_1, \dots, t_n\}$ . The solution of  $E$  for  $x_i \in \text{Dom}(E)$  is the term in  $E^*(x_i) \in \mathcal{T}(\Sigma_{\perp}, \mathcal{V})$  the defined by:*

$$E^*(x_i)(p) = \begin{cases} t_i(p) & \text{if } p \in \text{Pos}(t_i) \text{ and } t_i(p) \notin \text{Dom}(E) \\ \perp & \text{if } t_i(p) = x_j \in \text{Dom}(E) \text{ looping} \\ E^*(x_j)(q) & \text{if } t_i(p) = x_j \in \text{Dom}(E) \text{ non-looping with } p = p' \cdot q \\ \text{undefined} & \text{otherwise} \end{cases}$$

*The pair  $\langle E, x_i \rangle$ , or  $E_{x_i}$  for short, is said to represent a term  $t$  (or is a regular representation of  $t$ ) if  $E^*(x_i) = t$ .*

Below, we often write  $E(x_i)$  for  $t_i$ , given a regular system of equations  $E = \{x_1 = t_1, \dots, x_n = t_n\}$ .

The above definition extends the usual definition of a regular system of equations by not restricting all variables in the domain to be non-looping. Remark that a non-looping regular system (over  $\Sigma_{\perp}$ ) can be obtained by replacing every equation  $x = t$  with  $x$  looping by  $x = \perp$ .

As in [1], the extension with  $\perp$  allows for the treatment of collapsing rules, i.e., rules with a variable on their right-hand side, as all other rules. No special measures need to be taken to handle the looping that may be introduced by collapsing rules (see also Example 4.2).

Remark that each  $E^*$  can be regarded as a substitution and, hence, can be extended to  $E^* : \mathcal{T}_{reg}(\Sigma_{\perp}, \mathcal{V}) \rightarrow \mathcal{T}_{reg}(\Sigma_{\perp}, \mathcal{V})$  in the usual way. By definition, we have  $E^*(x) = E^*(t)$  for every  $x = t \in E$ .

From here onwards, we assume that  $\Sigma$  always includes  $\perp$ . We now have the following.

**Lemma 3.2.** *Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  be a regular system and let  $s_i$  be a finite term with  $s_i \notin \text{Dom}(E)$  and  $E^*(x_i) = E^*(s_i)$ . If  $F$  is identical to  $E$  but with  $x_i = t_i$  replaced by  $x_i = s_i$ , then  $E^* = F^*$ .*

**Lemma 3.3.** *Let  $E$  and  $F$  be regular systems and suppose there exists a surjection  $\delta : \text{Dom}(E) \rightarrow \text{Dom}(F)$  such that  $\delta(y) = \delta(s) \in F$  for every  $y = s \in E$ , where  $\delta$  is extended to  $\delta : \mathcal{T}_{fin}(\Sigma, \mathcal{V}) \rightarrow \mathcal{T}_{fin}(\Sigma, \mathcal{V})$  in the usual way. Then,  $E^*(y) = F^*(\delta(y))$  for every  $y \in \text{Dom}(E)$ .*

Surjectivity is required in the above lemma: Consider  $E = \{x = f(y)\}$  and  $F = \{x = f(y), y = a\}$ . If  $\delta$  is the (non-surjective) identity function, then  $E^*(x) = f(y)$ , while  $F^*(x) = f(a)$ .

**Subterm Positions.** For each  $x \in \text{Dom}(E)$ , let  $\mathcal{U}_E(x)$  be the smallest set satisfying (1)  $x \in \mathcal{U}_E(x)$  and (2) if  $y \in \mathcal{U}_E(x)$  and  $y = t \in E$ , then  $\mathcal{V}(t) \cap \text{Dom}(E) \subseteq \mathcal{U}_E(x)$ . We write  $y \sqsubseteq_E x$  if  $y \in \mathcal{U}_E(x)$ . It is readily checked that  $\sqsubseteq_E$  is transitive. The subscript  $E$  is omitted if it is obvious from the context. We write  $W \sqsubseteq x$  if  $y \sqsubseteq x$  for all  $x \in W$ . For  $x \in \text{Dom}(E)$ , let  $E \upharpoonright x = \{y = t \in E \mid y \sqsubseteq x\}$ . Obviously,  $E \upharpoonright x$  contains all equations necessary to define  $E^*(x)$ . Moreover, we have  $E^*(y) = (E \upharpoonright x)^*(y)$  for each  $y \sqsubseteq x$ .

For every  $y \sqsubseteq x$ , there is a set of positions in the rational term  $E^*(x)$  corresponding to occurrences of  $E^*(y)$ . Such a set of positions, called a set of subterm positions, is defined next. The notion will be heavily used in our proofs.

**Definition 3.4 (Subterm Positions).** Let  $E$  be a regular system. For each  $x, y \in \text{Dom}(E)$  such that  $y \sqsubseteq x$ , the set  $\mathcal{SP}_{E_x}(y)$  of subterm positions is the smallest set satisfying: (1)  $\epsilon \in \mathcal{SP}_{E_x}(x)$  and (2)  $p \cdot q \in \mathcal{SP}_{E_x}(y)$  if  $p \in \mathcal{SP}_{E_x}(z)$  and there exists an equation  $z = t \in E$  such that  $t \upharpoonright_q = y$ .

For a set  $W \subseteq \text{Dom}(E)$ , we put  $\mathcal{SP}_{E_x}(W) = \bigcup_{y \in W} \mathcal{SP}_{E_x}(y)$ .

*Example 3.5 (Subterm Positions).* Let  $E = \{x = f(y), y = g(x)\}$ . We have  $\mathcal{SP}_{E_x}(x) = \{\epsilon\} \cup \{p \cdot 1 \mid p \in \mathcal{SP}_{E_x}(y)\}$  and  $\mathcal{SP}_{E_x}(y) = \{p \cdot 1 \mid p \in \mathcal{SP}_{E_x}(x)\}$ . Hence,  $\mathcal{SP}_{E_x}(x) = \{1^{2n} \mid n \geq 0\}$  and  $\mathcal{SP}_{E_x}(y) = \{1^{2n+1} \mid n \geq 0\}$ .

The following is proved in a straightforward way.

**Proposition 3.6.** Let  $E$  be a regular system and  $x \in \text{Dom}(E)$ .

1. If  $W \sqsubseteq x$ , then  $\mathcal{SP}_{E_x}(W)$  is regular.
2. If  $\text{Ran}(E) \cap \text{Dom}(E) = \emptyset$  and  $p < q$  such that  $p \in \mathcal{SP}_{E_x}(y)$  and  $q \in \mathcal{SP}_{E_x}(z)$  with  $y \neq z$ , then  $q \notin \{p \cdot p' \mid p' \in \text{Pos}_\Sigma(E(y))\}$ .

**Canonical Systems.** A regular system  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  is said to be *canonical* if for every  $1 \leq i \leq n$ , either (1)  $t_i \in \mathcal{V} \setminus \text{Dom}(E)$  or (2)  $t_i = f(y_1, \dots, y_m)$  for some  $f \in \Sigma$  and  $y_1, \dots, y_m \in \text{Dom}(E)$  [7]. A regular representation  $E_x$  is said to be canonical if  $E$  is so. Canonical regular systems have the following properties.

**Proposition 3.7.** Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  be a canonical regular system.

1. If  $E^*(x) = s$ , then for every  $p \in \text{Pos}(s)$  there exists a variable  $y \sqsubseteq x$  such that  $s \upharpoonright_p = E^*(y)$ .
2. For all  $1 \leq i \leq n$ , if  $s_i$  is a finite term such that  $E^*(x_i) = s_i \sigma$  for some substitution  $\sigma$ , then a variable substitution  $\rho$  exists such that  $E^*(x_i) = E^*(s_i \rho)$ .

The next proposition can be proved in a straightforward way.

**Proposition 3.8 (Canonization).** Given a regular system  $E$ , a canonical regular system  $F$  can be constructed effectively such that (1)  $\text{Dom}(E) \subseteq \text{Dom}(F)$ , (2)  $E^*(x) = F^*(x)$  for every  $x \in \text{Dom}(E)$ , and (3)  $\mathcal{SP}_{E_x}(y) = \mathcal{SP}_{F_x}(y)$  for every  $x, y \in \text{Dom}(E)$  with  $y \sqsubseteq x$ .

A canonical regular system  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  is said to be *minimal* if  $E^*(x_i) \neq E^*(x_j)$  for every  $1 \leq i < j \leq n$ . A canonical regular representation  $E_x$  is minimal if so is  $E$ . We have the following.

**Proposition 3.9.** *Every rational term has a minimal canonical regular representation.*

The next proposition is a consequence of regularity.

**Proposition 3.10.** *Let  $\Delta$  be a regular set of positions of a rational term  $t$ . A canonical regular representation  $E_x$  of  $t$  and a set  $W \subseteq \text{Dom}(E)$  can be constructed effectively such that  $\Delta = \mathcal{SP}_{E_x}(W)$ .*

## 4 Rational Rewriting

A *rewrite rule* is a pair  $(l, r)$  terms over  $\Sigma$ , invariably written  $l \rightarrow r$  such that (1)  $l$  and  $r$  are finite, (2)  $l \in \mathcal{T}(\Sigma \setminus \{\perp\}, \mathcal{V})$  and  $l \notin \mathcal{V}$ , and (3)  $\mathcal{V}(r) \subseteq \mathcal{V}(l)$ . A *term rewriting system (TRS)* is a finite set of rewrite rules. We define rational rewrite steps.

**Definition 4.1 (Rational Rewrite Steps).** *Let  $s$  and  $t$  be rational terms. Define  $s \rightarrow_{l \rightarrow r} t$  if there exist regular representations  $E_x$  and  $F_x$  of  $s$  and  $t$ , resp., such that  $\text{Dom}(E) = \text{Dom}(F)$  and  $W \subseteq \text{Dom}(E)$  with (1)  $E(y) = F(y)$  for all  $y \in \text{Dom}(E) \setminus W$  and (2) for each  $y \in W$ , there exists a variable substitution  $\rho$  such that  $E(y) = l\rho$  and  $F(y) = r\rho$ .*

We write  $s \rightarrow_{l \rightarrow r}^\Delta t$  for  $\Delta = \mathcal{SP}_{E_x}(W)$ . Moreover, if  $\mathcal{R}$  is a TRS, we write  $s \rightarrow_{\mathcal{R}} t$  if there exists  $l \rightarrow r \in \mathcal{R}$  such that  $s \rightarrow_{l \rightarrow r} t$ .

*Example 4.2.* Let  $\mathcal{R} = \{f(x) \rightarrow x\}$  be a TRS and  $s = f(f(\dots f(\dots) \dots))$ .

1. Since  $E = \{x = f(x)\}$  and  $F = \{x = x\}$  are regular representations of  $s$  and  $\perp$ , resp., we have  $s \rightarrow_{\mathcal{R}}^\Delta \perp$ , where  $\Delta = \mathcal{SP}_{E_x}(x) = \{1^n \mid n \geq 0\}$ .
2. Since  $E = \{x = f(y), y = f(x)\}$  and  $F = \{x = f(y), y = x\}$  are regular representations of  $s$ , we have  $s \rightarrow_{\mathcal{R}}^\Delta s$  with  $\Delta = \mathcal{SP}_{E_x}(y) = \{1^{2n+1} \mid n \geq 0\}$ .
3. Since  $E = \{x = f(y), y = f(y)\}$  and  $F = \{x = f(y), y = y\}$  are regular representations of  $s$  and  $f(\perp)$ , resp., we have  $s \rightarrow_{\mathcal{R}}^\Delta f(\perp)$  with  $\Delta = \mathcal{SP}_{E_x}(y) = \{1 \cdot 1^n \mid n \geq 0\}$ .

*Remark 4.3.* Contrary to our definition, the definition of rational rewriting as presented in [6] does not require one to explicitly specify the employed rewrite rule. In fact, several rewrite rules may be used in a *single* rewrite step, as long as the set of positions at which the contracted redexes occur is regular. In [6], specifying a regular set of positions and no rewrite rules suffices, as orthogonality is assumed there throughout.

As we do not restrict ourselves to orthogonal systems, specifying just a rational set of positions does not suffice in our case. Consider, e.g.,  $\mathcal{R} = \{\alpha : f(x) \rightarrow g(x), \beta : f(x) \rightarrow h(x)\}$ ,  $s = f(f(\dots f(\dots) \dots))$ , and  $\Delta = \{1^n \mid n \geq 0\}$ . We can now choose  $\Psi = \{(\epsilon, \alpha), (1, \beta), (1^2, \beta), (1^3, \alpha), (1^4, \beta), (1^5, \beta), (1^6, \beta), \dots\}$ . Hence,  $s$  rewrites to  $g(h^2(g(h^3(\dots g(h^n(\dots))))))$ , which is clearly not rational.

*Remark 4.4.* Another difference between our work and [6] surfaces when we consider rules that are not left-linear. Consider, e.g.,  $\mathcal{R} = \{f(x, x) \rightarrow g(x)\}$ ,  $E = \{x = f(y, y), y = g(x)\}$ ,  $s = E^*(x)$  and  $\Delta = \{1^{2^n} \mid n \geq 0\}$ . Rewriting any redex at a position in  $\Delta$  is problematic, as it causes any redex at a prefix position to cease to exist.

Employing the definition from [6], the above problematic rewrites would be allowed — albeit falling outside the framework, as  $\mathcal{R}$  is not orthogonal. On the other hand, we do not have  $s \xrightarrow{l \rightarrow r}^{\Delta} t$  for any  $t$ : Suppose a regular representation  $E$  and a set  $W \subseteq \text{Dom}(E)$  exist such that  $\mathcal{SP}_{E_x}(W) = \Delta$  and such that for all  $y \in W$  we have  $E(y) = f(x', x')\rho_y$  with  $\rho_y$  a variable substitution. Consider any  $y \in W$  and suppose  $\rho_y(x') = z$ . By definition of  $\Delta$ , there is a  $v \in W$  such that  $\mathcal{SP}_{E_z}(v)$  is non-empty. However, since the same variable occurs twice in  $f(x', x')$ ,  $\mathcal{SP}_{E_z}(v)$  contains both positions of the form  $1 \cdot p$  and of the form  $2 \cdot p$ , contradicting the assumption that  $\mathcal{SP}_{E_x}(W)$  contains only positions with 1s.

**Rational Patterns.** As witnessed by Example 4.2, there are in general many ways to rewrite a rational term. The notion of a rational pattern is helpful to characterize each rewrite step uniquely. The definition of this notion, uses a labeling of rational terms.

Let  $\Sigma$  be a signature. We define a *marked signature* as  $\Sigma^\bullet = \Sigma \cup \{f^\bullet \mid f \in \Sigma\}$ , where for every  $f \in \Sigma$  the arity of  $f^\bullet$  is equal to the arity of  $f$ . The notion of a marked signature originates from [11].

**Definition 4.5 (Labeling).** Let  $t \in \mathcal{T}(\Sigma, \mathcal{V})$  and  $\Delta \subseteq \text{Pos}_\Sigma(t)$ . The labeled term  $\text{lab}(\Delta, t)$  over the signature  $\Sigma^\bullet$  is defined by

$$\text{lab}(\Delta, t)(p) = \begin{cases} t(p)^\bullet & \text{if } p \in \Delta \\ t(p) & \text{if } p \notin \Delta \end{cases}$$

**Lemma 4.6.** If  $t$  is a rational term and  $\Delta \subseteq \text{Pos}_\Sigma(t)$  a regular set of positions, then  $\text{lab}(\Delta, t)$  is rational and a minimal canonical representation  $E_x$  of  $\text{lab}(\Delta, t)$  and a set  $W \subseteq \text{Dom}(E)$  can be constructed effectively such that  $\Delta = \mathcal{SP}_{E_x}(W)$ .

Denoting  $\text{lab}(\{\epsilon\}, l)$  by  $l^\bullet$ , we define the following.

**Definition 4.7 (Rational Pattern).** Let  $t$  be a rational term. A pair  $\langle l, \Delta \rangle$  with  $l$  the left-hand side of a rewrite rule and  $\Delta \subseteq \text{Pos}_\Sigma(t)$  is a rational pattern in  $t$  if (1)  $\Delta$  is a regular set of positions in  $t$ , and (2) for any  $p \in \Delta$ ,  $\text{lab}(\Delta, t)|_p = l^\bullet \sigma$  for some substitution  $\sigma$ .

*Example 4.8.* If  $E = \{x = f(y, y), y = g(y)\}$  and  $s = E^*(x)$ , then  $\langle f(x, g(x)), \{\epsilon\} \rangle$  is a rational pattern in  $s$ . Moreover, if  $\Delta = \{1 \cdot 1^n \mid n \geq 0\}$ , then  $\langle g(x), \Delta \rangle$  is a rational pattern in  $s$ , while  $\langle g(g(x)), \Delta \rangle$  is not.

If  $F = \{x = f(y, z), y = g(y), z = g(h(z))\}$  and  $t = F^*(x)$ , then  $\langle g(x), \Lambda \rangle$  with  $\Lambda = \{1 \cdot 1^n \mid n \geq 0\} \cup \{2 \cdot 1^{2^n} \mid n \geq 0\}$  is a rational pattern in  $t$ . However, if  $G = \{x = f(y, y), y = g(x)\}$ ,  $u = G^*(x)$  and  $\Gamma = \{1^{2^n} \mid n \geq 0\}$ , then  $\langle f(x, x), \Gamma \rangle$  is not a rational pattern in  $u$  (see also Remark 4.4).

To check the second condition in the definition of a rational pattern, the following necessary and sufficient condition is useful, where  $\Delta|_p = \{q \mid p \cdot q \in \Delta\}$ .

**Lemma 4.9.** *If  $t$  is a rational term,  $\Delta \subseteq \mathcal{Pos}_\Sigma(t)$ , and  $l$  is the left-hand side of a rewrite rule, then  $\text{lab}(\Delta, t) = l \bullet \sigma$  for a substitution  $\sigma$  iff (1)  $t = l\sigma'$  for a substitution  $\sigma'$ , (2)  $\Delta \cap \mathcal{Pos}_\Sigma(l) = \{\epsilon\}$ , and (3)  $\Delta|_p = \Delta|_q$  for any  $p, q \in \mathcal{Pos}_\Sigma(l)$  with  $l(p) = l(q)$ .*

The main result of this section is the correspondence between rational rewrite steps and rational patterns. To prove, we introduce two notions: propagation and independence.

**Definition 4.10 (Propagation).** *Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  be a regular system and  $s$  a finite term.*

1.  $E$  propagates to a regular system  $F$  for variables  $x_i$  and  $x_k$ , denoted  $E \gg_k^i F$ , if  $t_i = C[x_k]$  in  $E$  and  $F = \{x_j = t_j \in E \mid j \neq i\} \cup \{x_i = C[t_k]\}$ .
2.  $s$  propagates to a finite term  $t$  under  $E$ , denoted  $s >_E t$ , if  $s = C[x_i]$  and  $t = C[t_i]$  for  $x_i = t_i \in E$ . The reflexive, transitive closure of  $>_E$  is  $\gg_E$ .

Observe that  $s \gg_E t$  implies  $E^*(s) = E^*(t)$ .

*Example 4.11.* Let  $E = \{x = f(y, z), y = g(z), z = h(y)\}$ . We have

$$\begin{aligned} E \gg_2^1 \{x = f(g(z), z), y = g(z), z = h(y)\} \\ \gg_3^1 \{x = f(g(z), h(y)), y = g(z), z = h(y)\}. \end{aligned}$$

Moreover,

$$f(y, z) >_E f(g(z), z) >_E f(g(z), h(y)) >_E f(g(h(y)), h(y)).$$

Hence,  $f(y, z) \gg_E f(g(h(y)), h(y))$ .

**Lemma 4.12.** *Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  be a minimal canonical regular system. If  $s$  is a finite term and  $E^*(s) = E^*(x_i)$  with  $s \neq x_i$ , then  $t_i \gg_E s$ .*

**Definition 4.13 (Independence).** *Let  $\Delta$  be a set of positions. If  $l$  is the left-hand side of a rewrite rule, then  $\Delta$  is  $l$ -independent if  $p \cdot q \notin \Delta$  for any  $p \in \Delta$  and  $q \in \mathcal{Pos}_\Sigma(l) \setminus \{\epsilon\}$ .*

**Lemma 4.14.** *Let  $l$  be the left-hand side of a rewrite rule,  $s$  a rational term,  $\Delta \subseteq \mathcal{Pos}_\Sigma(s)$ , and  $p \in \Delta$ . If there exists a substitution  $\sigma$  such that  $\text{lab}(\Delta, s)|_p = l \bullet \sigma$ , then  $\Delta$  is  $l$ -independent.*

**Lemma 4.15.** *Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$  be a canonical regular system and  $x \in \text{Dom}(E)$ . Suppose  $l$  is the left-hand side of a rewrite rule and  $\mathcal{SP}_{E_x}(W)$  is  $l$ -independent for  $W \sqsubseteq x$ . Moreover, suppose that for each  $x_i \in W$  there exists a variable substitution  $\rho_i$  with  $t_i \gg_E l\rho_i$ . If  $F = \{x_i = t_i \in E \mid x_i \notin W\} \cup \{x_i = l\rho_i \mid x_i \in W\}$ , then (1)  $E \gg_{k_1}^{i_1} \dots \gg_{k_m}^{i_m} F$  for some  $x_{i_1}, \dots, x_{i_m} \in W$  and  $x_{k_1}, \dots, x_{k_m} \notin W$ , (2)  $E^* = F^*$ , and (3)  $\mathcal{SP}_{E_x}(x_i) = \mathcal{SP}_{F_x}(x_i)$  for any  $x_i \in W$ .*

Rational patterns and rational rewrite steps are related as follows.

**Lemma 4.16 (Correspondence).** *If  $s$  is a rational term,  $l \rightarrow r$  a rewrite rule, and  $\Delta \subseteq \text{Pos}(s)$ , then  $s \rightarrow_{l \rightarrow r}^\Delta t$  for some  $t$  iff  $\langle l, \Delta \rangle$  is a rational pattern in  $s$ .*

*Proof.* ( $\Rightarrow$ ) By definition, there exist regular representations  $E_x$  and  $F_x$  of  $s$  and  $t$ , resp., such that  $\text{Dom}(E) = \text{Dom}(F)$  and  $W \subseteq \text{Dom}(E)$  with (1)  $\Delta = \mathcal{SP}_{E_x}(W)$ , (2)  $E(y) = F(y)$  for all  $y \in \text{Dom}(E) \setminus W$ , and (3) for each  $y \in W$ ,  $E(y) = l\rho$  and  $F(y) = r\rho$  for some variable substitution  $\rho$ .

Without loss of generality we may assume  $\text{Dom}(E) \cap \text{Ran}(E) = \emptyset$ . We show that  $\langle l, \Delta \rangle$  is a rational pattern in  $s$ . First,  $\Delta$  is regular by Proposition 3.6(1). Moreover, as  $l$  is the left-hand side of a rewrite rule, we have  $\Delta \subseteq \text{Pos}_\Sigma(t)$ . Let  $p \in \Delta$ , then  $\text{lab}(\Delta, t)|_p = \text{lab}(\Delta|_p, t|_p)$ . To show that  $\text{lab}(\Delta|_p, t|_p) = l^\bullet \sigma$  for some  $\sigma$ , we use Lemma 4.9.

1. As  $p \in \Delta$ , there is  $y \in W$  such that  $p \in \mathcal{SP}_{E_x}(y)$  and  $E(y) = l\rho$  for a variable substitution  $\rho$ . If  $\sigma' = E^* \circ \rho$ , then  $s|_p = E^*(y) = E^*(l\rho) = (E^* \circ \rho)(l) = l\sigma'$ .
2. Obviously,  $\epsilon \in \Delta|_p$ , as  $p \in \Delta$ . Suppose there exists a  $q \in \text{Pos}_\Sigma(l) \setminus \{\epsilon\}$  such that  $p \cdot q \in \Delta$ . Then, there are  $y, z \in W$  with  $y \neq z$  such that  $p \in \mathcal{SP}_{E_x}(y)$  and  $p \cdot q \in \mathcal{SP}_{E_x}(z)$ . Since  $q \neq \epsilon$ , we have  $p < q$  and by Proposition 3.6(2),  $p \cdot q \notin \{p \cdot p' \mid p' \in \text{Pos}_\Sigma(l\rho)\}$ , where  $l\rho = E(z)$ , contradicting  $q \in \text{Pos}_\Sigma(l)$ . Hence,  $\Delta|_p \cap \text{Pos}_\Sigma(l) = \{\epsilon\}$ .
3. We have  $p \in \mathcal{SP}_{E_x}(y)$  for some  $y \in W$ , as  $p \in \Delta$ . Suppose  $p_1, p_2 \in \text{Pos}_\nu(l)$  and  $l(p_1) = l(p_2) = z$ . As  $y = l\rho \in E$  for some variable substitution  $\rho$ , we have  $l\rho(p_1) = l\rho(p_2) = \rho(z)$ . If  $\rho(z) \notin \text{Dom}(E)$ , then  $\Delta|_{p \cdot p_1} = \emptyset = \Delta|_{p \cdot p_2}$ . If  $\rho(z) \in \text{Dom}(E)$  then  $\rho(z) \sqsubseteq y$  and, hence,  $\Delta|_{p \cdot p_1} = \mathcal{SP}_{E_{\rho(z)}}(W) = \Delta|_{p \cdot p_2}$ .

( $\Leftarrow$ ) By assumption,  $\Delta \subseteq \text{Pos}_\Sigma(s)$  is regular. Hence, by Lemma 4.6, there exists a minimal canonical representation  $E_x$  of  $\text{lab}(\Delta, s)$  and a set  $W \subseteq \text{Dom}(E)$  such that  $\Delta = \mathcal{SP}_{E_x}(W)$ . Let  $E = \{x_1 = t_1, \dots, x_n = t_n\}$ . By assumption, for every  $p \in \Delta$ , there exists a substitution  $\sigma$  such that  $\text{lab}(\Delta, s)|_p = l^\bullet \sigma$ . Hence, by Lemma 4.14,  $\Delta$  is  $l$ -independent. Furthermore, for each  $x \in \mathcal{V}(l)$ ,  $\sigma(x)$  is a subterm of  $\text{lab}(\Delta, s)$ . Since  $E$  is minimal and canonical, we have for each  $\sigma(x)$  with  $x \in \mathcal{V}(l)$  that there exists unique  $x_j \in \text{Dom}(E)$  such that  $\sigma(x) = E^*(x_j)$ . Thus, for each  $x_i \in W$ , there exists a variable substitution  $\rho_i$  such that  $E^*(x_i) = \text{lab}(\Delta, s)|_p = l^\bullet \sigma = E^*(l^\bullet \rho_i)$ . Then, by Lemma 4.12, we obtain  $t_i \gg_E l^\bullet \rho_i$ . Let  $\hat{E} = \{x_i = t_i \in E \mid x_i \notin W\} \cup \{x_i = l^\bullet \rho_i \mid x_i \in W\}$ . By Lemma 4.15,  $E^*(x) = \hat{E}^*(x)$  and  $\Delta = \mathcal{SP}_{E_x}(W) = \mathcal{SP}_{\hat{E}_x}(W)$ . Let  $\tilde{E}$  be the regular system obtained from  $\hat{E}$  by removing all marks. Then,  $s = \tilde{E}^*(x)$  and  $s \rightarrow_{l \rightarrow r}^\Delta F^*(x)$  with  $F = \{x_i = t_i \in \tilde{E} \mid x_i \notin W\} \cup \{x_i = r\rho_i \mid x_i \in W\}$ .  $\square$

Our characterization of rewrite steps by rational patterns leads us to obtain following decidability result for rewrite steps.

**Lemma 4.17 (Constructiveness).** *Let  $s$  be a rational term,  $\Delta$  a regular set of positions, and  $l \rightarrow r$  a rewrite rule.*

1. *It is decidable whether  $\langle l, \Delta \rangle$  is a rational pattern in  $s$ .*

2. If  $\langle l, \Delta \rangle$  is a rational pattern in  $s$ , then a regular representation of a rational term  $t$  can be constructed effectively such that  $s \rightarrow_{l \rightarrow r}^{\Delta} t$ .

*Proof.* (1) Let  $s$  be a rational term and  $\Delta$  a regular set of positions (possibly with  $\Delta \not\subseteq \text{Pos}(s)$ ). We can check whether  $\Delta \subseteq \text{Pos}(s)$ , as both  $\Delta$  and  $\text{Pos}(s)$  are regular. If  $\Delta \subseteq \text{Pos}(s)$  does not hold, then return *no*. By Lemma 4.6,  $\text{lab}(\Delta, s)$  is regular and its regular representation  $E_x$  such that  $\Delta = \mathcal{SP}_{E_x}(W)$  for some  $W \subseteq \text{Dom}(E)$  can be constructed effectively. For each  $y \in W$  check whether  $E^*(y) = l \bullet \sigma_y$  for some substitution  $\sigma_y$ . This check is decidable, as the matching problem for rational terms is decidable [7]. (2) As above, construct the set  $W$  and the substitution  $\sigma_y$  such that  $E^*(y) = l \bullet \sigma_y$  for each  $y \in W$ . Next, define  $F = \{y = s \in E \mid y \notin W\} \cup \{y = r\sigma_y \in E \mid y \in W\}$ . Then,  $F_x$  is a regular representation of  $t$  such that  $s \rightarrow_{l \rightarrow r}^{\Delta} t$ .  $\square$

## 5 Decidability and Constructive Confluence

The proofs in this section heavily depend on the notion of a product:

**Definition 5.1.** Let  $s \in \mathcal{T}(\Sigma, \mathcal{V})$  and  $t \in \mathcal{T}(\Gamma, \mathcal{V})$  be terms. The terms are similar, denoted  $s \sim t$ , if  $\text{Pos}_{\mathcal{V}}(s) = \text{Pos}_{\mathcal{V}}(t)$  and  $\text{Pos}_{\Sigma}(s) = \text{Pos}_{\Gamma}(t)$ .

Let  $s$  and  $t$  be similar. The product  $s \times t \in \mathcal{T}(\Sigma \times \Gamma, \mathcal{V})$  of  $s$  and  $t$  is  $(s \times t)(p) = \langle s(p), t(p) \rangle$  for all  $p$ , where tuples  $\langle x, y \rangle$  are considered to be variables. Moreover, let  $E$  and  $F$  be canonical regular systems over, resp.,  $\Sigma$  and  $\Gamma$ . The product  $E \times F$  over  $\Sigma \times \Gamma$  is

$$E \times F = \{ \langle x, y \rangle = s \times t \mid x = s \in E, y = t \in F, s \sim t \}.$$

Let  $t \in \mathcal{T}(\Sigma \times \Gamma, \mathcal{V})$ . The projections  $\pi_1$  and  $\pi_2$  to  $\Sigma$  and  $\Gamma$ , resp., are defined by

$$\pi_i(t)(p) = \begin{cases} f_i & \text{if } t(p) = \langle f_1, f_2 \rangle \in \Sigma \times \Gamma \\ x & \text{if } t(p) = x \in \mathcal{V} \end{cases}$$

where  $i \in \{1, 2\}$ . Moreover, let  $E$  be a canonical regular system over  $\Sigma \times \Gamma$ . The projections  $\pi_1$  and  $\pi_2$  to  $\Sigma$  and  $\Gamma$ , resp., are defined by

$$\pi_i(E) = \{ \pi_i(x) = \pi_i(t) \mid x = t \in E \},$$

where  $i \in \{1, 2\}$ .

The following properties of products are proved in a straightforward way.

**Proposition 5.2.** Let  $s$  and  $t$  be similar rational terms and let  $E_x$  and  $F_y$  be canonical regular representations of, resp.,  $s$  and  $t$ . If  $z = \langle x, y \rangle$  and  $G = E \times F$ , then for products:

1.  $G^*(\langle x', y' \rangle) = E^*(x') \times F^*(y')$  for  $\langle x', y' \rangle \sqsubseteq z$ ;
2.  $\mathcal{SP}_G(\langle x', y' \rangle) = \mathcal{SP}_{E_x}(x') \cap \mathcal{SP}_{F_y}(y')$ .

Moreover, for projections:

3.  $(\pi_1(G))^*(\langle x', y' \rangle) = E^*(x')$  and  $(\pi_2(G))^*(\langle x', y' \rangle) = F^*(y')$  for  $\langle x', y' \rangle \sqsubseteq z$ ;
4.  $\mathcal{SP}_{E_x}(x') = \mathcal{SP}_{G_z}(\{x'\} \times \text{Dom}(F))$  and  $\mathcal{SP}_{F_y}(y') = \mathcal{SP}_{G_z}(\text{Dom}(E) \times \{y'\})$ .

The next lemma, which will be used below, follows from Proposition 5.2(2).

**Lemma 5.3.** *Let  $E_x$  and  $F_y$  be canonical regular representations of a term  $s$  and let  $V \sqsubseteq_E x$  and  $W \sqsubseteq_F y$  with  $\mathcal{SP}_{E_x}(V) = \mathcal{SP}_{F_y}(W) = \Delta$ .*

1. If  $U = \{\langle x', y' \rangle \sqsubseteq_{E \times F} \langle x, y \rangle \mid x' \in V, y' \in W\}$ , then  $\mathcal{SP}_{(E \times F)(x, y)}(U) = \Delta$ .
2.  $\{\langle x', y' \rangle \sqsubseteq_{E \times F} \langle x, y \rangle \mid x' \in V, y' \notin W\} = \emptyset$  and  $\{\langle x', y' \rangle \sqsubseteq_{E \times F} \langle x, y \rangle \mid x' \notin V, y' \in W\} = \emptyset$ .

The next lemma shows that a rewrite step  $s \rightarrow_{l \rightarrow r}^\Delta t$  is uniquely defined by a term  $s$ , a regular set  $\Delta$  of positions in  $s$ , and a rewrite rule  $l \rightarrow r$ .

**Lemma 5.4.** *If  $s \rightarrow_{l \rightarrow r}^\Delta t_1$  and  $s \rightarrow_{l \rightarrow r}^\Delta t_2$ , then  $t_1 = t_2$ .*

*Proof.* By assumption, there are representations  $E_x$  and  $F_y$  of  $s$  and representations  $E'_x$  and  $F'_y$  of  $t_1$  and  $t_2$ , resp., with  $V \sqsubseteq x$  and  $W \sqsubseteq y$  such that  $\Delta = \mathcal{SP}_{E_x}(V) = \mathcal{SP}_{F_y}(W)$  and  $x_i = l\rho_i$  for all  $x_i \in V$  and  $y_j = l\delta_j$  for all  $y_j \in W$  with  $\rho_i$  and  $\delta_j$  variable substitutions. Without loss of generality, we may assume that  $E$  and  $E'$ , resp.  $F$  and  $F'$ , are canonical with exception of the equations with a variable from  $V$ , resp.  $W$ , on the left-hand side. Apply Proposition 3.8 to obtain canonical representations  $\hat{E}_x$  and  $\hat{F}_y$ . Since  $E^*(x) = F^*(y) = s$ , we have that  $(\hat{E} \times \hat{F})_{(x, y)}$  is well-defined and a regular representation of  $s \times s$ .

Let  $U = \{\langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle \mid x_i \in V, y_j \in W\}$  and  $\hat{G} = \pi_1(\hat{E} \times \hat{F}) = \pi_2(\hat{E} \times \hat{F})$ . By Lemma 5.3(1),  $\mathcal{SP}_{(\hat{E} \times \hat{F})_{(x, y)}}(U) = \Delta$  and, by Proposition 5.2(3),  $\hat{G}^*(\langle x_i, y_j \rangle) = \hat{E}^*(x_i) = \hat{F}^*(y_j)$ , for any  $\langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle$ . Furthermore, for every  $\langle x_i, y_j \rangle \in U$ , we have  $E^*(x_i) = E^*(l\rho_i)$  and  $F^*(y_j) = F^*(l\delta_j)$ . Hence, by Proposition 5.2(1), for every  $\langle x_i, y_j \rangle \in U$ , we have  $(E \times F)^*(\langle x_i, y_j \rangle) = E^*(x_i) \times F^*(y_j) = E^*(l\rho_i) \times F^*(l\delta_j) = (E \times F)^*((l \times l)\xi_{i, j})$  where  $\xi_{i, j} = \{\langle z, z \rangle := \langle \rho_i(z), \delta_j(z) \rangle \mid z \in \mathcal{V}(l)\}$ . Hence,  $\hat{G}^*(\langle x_i, y_j \rangle) = \pi_1((E \times F)^*(\langle x_i, y_j \rangle)) = \pi_1((E \times F)^*((l \times l)\xi_{i, j})) = \hat{G}^*(l\xi_{i, j})$ . And, by Lemma 3.2,  $\hat{G}^* = G^*$  where  $G = \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin U\} \cup \{\langle x_i, y_j \rangle = l\xi_{i, j} \mid \langle x_i, y_j \rangle \in U\}$ .

Let  $G' = \{\langle x_i, y_j \rangle = w \in G \mid \langle x_i, y_j \rangle \notin U\} \cup \{\langle x_i, y_j \rangle = r\xi_{i, j} \mid \langle x_i, y_j \rangle \in U, \langle x_i, y_j \rangle = l\xi_{i, j} \in G\}$ . Define  $\delta : \text{Dom}(G') \rightarrow \text{Dom}(E')$  as  $\delta(\langle x_i, y_j \rangle) = x_i$  and observe that  $\{\langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle \mid x_i \in V, y_j \notin W\}$  is empty by Lemma 5.3(2). Thus,  $\delta$  is surjective and for any  $\langle x_i, y_j \rangle = w \in G'$  we have  $\delta(\langle x_i, y_j \rangle) = \delta(w) \in E'$ . By Lemma 3.3, we now have  $G'^*(\langle x_i, y_j \rangle) = E'^*(x_i)$ . Similarly, we have  $G'^*(\langle x_i, y_j \rangle) = F'^*(y_j)$ . Thus,  $t_1 = E'^*(x) = G'^*(\langle x, y \rangle) = F'^*(y) = t_2$ .  $\square$

**Theorem 5.5.** *Let  $\mathcal{R}$  be a TRS.*

1. For a rational term  $s$  and regular set of positions  $\Delta$ , the number of rational terms  $t$  such that  $s \rightarrow_{\mathcal{R}}^\Delta t$  is finite and a regular representation of each term  $t$  can be constructed effectively.

2. For rational terms  $s$  and  $t$  and a regular set of positions  $\Delta$ , it is decidable whether  $s \rightarrow_{\mathcal{R}}^{\Delta} t$  or not.
3. For a given regular representation of rational term  $s$ , it is decidable whether  $s$  is in normal form with respect to  $\mathcal{R}$ . If not, a regular representation of a rational term  $t$  such that  $s \rightarrow_{\mathcal{R}} t$  can be constructed effectively.

*Proof.* (1) By Lemma 5.4, for a given  $s$ ,  $\Delta$ , and  $l \rightarrow r$  there is at most one  $t$  such that  $s \rightarrow_{l \rightarrow r}^{\Delta} t$ . Thus, the number of terms  $t$  such that  $s \rightarrow_{\mathcal{R}}^{\Delta} t$  is bounded by  $|\mathcal{R}|$ . Furthermore, we can effectively construct a term  $t$  with  $s \rightarrow_{l \rightarrow r}^{\Delta} t$  by Lemma 4.17 if there is such a term. (2) Immediately by the previous. (3) A rational term  $s$  is  $\mathcal{R}$ -normal if there is no subterm  $u$  of  $s$  with  $u = l\sigma$  for some  $l \rightarrow r \in \mathcal{R}$  and  $\sigma$ . Since the (finite) set of subterms of any rational term is effectively constructed, this is decidable. Take a canonical representation  $E_x$  of  $s$ . Then for any subterm  $s'$  of  $s$  there is variable  $y \in \text{Dom}(E)$  such that  $s' = E^*(y)$ . We can check whether  $E^*(y) = l\sigma$  for some rule  $l \rightarrow r \in \mathcal{R}$  and substitution  $\sigma$ . If there is such a rule and such a substitution, then there exists a variable substitution  $\rho$  such that  $E^*(y) = E^*(l\rho)$  by Proposition 3.7. Define  $E' = (E \setminus \{y = t\}) \cup \{y = l\rho\}$ . By Lemma 3.2,  $E'_x$  is a representation of  $s$ . Moreover, we can effectively construct  $F' = (E' \setminus \{y = l\rho\}) \cup \{y = r\rho\}$  and, hence,  $s \rightarrow_{\mathcal{R}} t$  for  $t = F'^*(x)$ .  $\square$

Let  $s$  and  $t$  be finite terms with  $\mathcal{V}(s) \cap \mathcal{V}(t) = \emptyset$ . Then,  $s$  overlaps  $t$  (at position  $p$ ) if there exists a non-variable subterm  $u = t|_p$  of  $t$  such that  $u$  and  $s$  are unifiable in  $\mathcal{T}_{\text{fin}}(\Sigma, \mathcal{V})$ . Let  $l_1 \rightarrow r_1$  and  $l_2 \rightarrow r_2$  be rewrite rules. Suppose  $l_1$  overlaps  $l_2$  at position  $p$  (without loss of generality  $\mathcal{V}(l_1) \cap \mathcal{V}(l_2) = \emptyset$ ). Let  $\sigma$  be the most general unifier of  $l_1$  and  $l_2|_p$ . The pair  $\langle l_2[r_1]_p\sigma, r_2\sigma \rangle$  is called a *critical pair*, where  $l_2[r_1]_p\sigma$  is the result of rewriting  $l_2\sigma$  by means of  $l_1 \rightarrow r_1$  at position  $p$ . In case of self-overlap (i.e., when  $l_1 \rightarrow r_1$  and  $l_2 \rightarrow r_2$  are identical modulo renaming of variables), we do not consider  $p = \epsilon$ . The critical pair  $\langle l_2[r_1]_p\sigma, r_2\sigma \rangle$  is *trivial* if  $l_2[r_1]_p\sigma = r_2\sigma$ . A TRS  $\mathcal{R}$  is *orthogonal* if it is left-linear and there is no critical pairs and *weakly orthogonal* if all critical pairs are trivial instead.

**Lemma 5.6 (Diamond Property).** *Let  $\mathcal{R}$  be an orthogonal TRS. For a rational term  $s$  and regular set of positions  $\Delta$  and  $\Gamma$  such that  $s \rightarrow_{\mathcal{R}}^{\Delta} t_1$  and  $s \rightarrow_{\mathcal{R}}^{\Gamma} t_2$ , a regular representation of a rational term  $u$  and regular sets of positions  $\Delta'$  and  $\Gamma'$  such that  $t_1 \rightarrow_{\mathcal{R}}^{\Delta'} u$  and  $t_2 \rightarrow_{\mathcal{R}}^{\Gamma'} u$  can be constructed effectively.*

*Proof.* By assumption, there are representations  $E_x$  and  $F_y$  of  $s$  and representations  $E'_x$  and  $F'_y$  of  $t_1$  and  $t_2$ , resp., with  $V \sqsubseteq x$  and  $W \sqsubseteq y$  such that  $\Delta = \mathcal{SP}_{E_x}(V)$  and  $\Gamma = \mathcal{SP}_{F_y}(W)$  and such that  $x_i = l_1\rho_i$  for a rule  $l_1 \rightarrow r_1$  and all  $x_i \in V$  and  $y_j = l_2\delta_j$  for a rule  $l_2 \rightarrow r_2$  and all  $y_j \in W$  with  $\rho_i$  and  $\delta_j$  variable substitutions. Apply Proposition 3.8 to obtain canonical representations  $\hat{E}_x$  and  $\hat{F}_y$ . Since  $E^*(x) = F^*(y) = s$ , we have that  $(\hat{E} \times \hat{F})_{\langle x, y \rangle}$  is well-defined and a regular representation of  $s \times s$ . By Proposition 5.2(4), we have  $\Delta = \mathcal{SP}_{\hat{E}_x}(V) = \mathcal{SP}_{(\hat{E} \times \hat{F})_{\langle x, y \rangle}}(V')$  with  $V' = \{\langle x_i, y_j \rangle \in V \times \text{Dom}(\hat{F}) \mid \langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle\}$ . Similarly,  $\Gamma = \mathcal{SP}_{\hat{F}_y}(W) = \mathcal{SP}_{(\hat{E} \times \hat{F})_{\langle x, y \rangle}}(W')$ , where  $W' = \{\langle x_i, y_j \rangle \in \text{Dom}(\hat{E}) \times W \mid \langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle\}$ .

Let  $\hat{G} = \pi_1(\hat{E} \times \hat{F}) = \pi_2(\hat{E} \times \hat{F})$ . Then,  $\Delta = \mathcal{SP}_{\hat{G}_{\langle x, y \rangle}}(V')$  and  $\Gamma = \mathcal{SP}_{\hat{G}_{\langle x, y \rangle}}(W')$ . Moreover, by Proposition 5.2(3), we have for any  $\langle x_i, y_j \rangle \sqsubseteq \langle x, y \rangle$

that  $\hat{G}^*(\langle x_i, y_j \rangle) = E^*(x_i) = F^*(y_j)$ ; in particular,  $\hat{G}_{\langle x, y \rangle}$  is a canonical regular representation of  $s$ . Furthermore, for every  $\langle x_i, y_j \rangle \in V'$ , we have  $\hat{E}^*(x_i) = \hat{E}^*(l_1 \rho_i)$  and, for every  $\langle x_i, y_j \rangle \in W'$ ,  $\hat{F}^*(y_j) = \hat{F}^*(l_2 \delta_j)$ . Hence, by Lemma 3.2,  $\hat{G}^* = G_1^*$  where  $G_1 = \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin V'\} \cup \{\langle x_i, y_j \rangle = l_1 \rho_{i,j} \mid \langle x_i, y_j \rangle \in V'\}$  and  $\rho_{i,j}$  is such that  $\hat{G}^*(\langle x_i, y_j \rangle) = G^*(l_1 \rho_{i,j})$ , where existence of  $\rho_{i,j}$  follows as  $\hat{G}$  is canonical and as every rewrite rule is left-linear. Similarly,  $\hat{G}^* = G_2^*$  where  $G_2 = \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin W'\} \cup \{\langle x_i, y_j \rangle = l_2 \delta_{i,j} \mid \langle x_i, y_j \rangle \in W'\}$  and  $\delta_{i,j}$  is such that  $\hat{G}^*(\langle x_i, y_j \rangle) = G^*(l_2 \delta_{i,j})$ .

Let  $G_1' = \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin V'\} \cup \{\langle x_i, y_j \rangle = r_1 \rho_{i,j} \mid \langle x_i, y_j \rangle \in V'\}$  and  $G_2' = \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin W'\} \cup \{\langle x_i, y_j \rangle = r_2 \delta_{i,j} \mid \langle x_i, y_j \rangle \in W'\}$ . Then,  $s = G_1^*(\langle x, y \rangle) \rightarrow^\Delta G_1'^*(\langle x, y \rangle)$  and, hence,  $t_1 = G_1'^*(\langle x, y \rangle)$  by Lemma 5.4. Similarly, we have  $t_2 = G_2'^*(\langle x, y \rangle)$ . We now distinguish two cases, where we observe that  $l_1$  and  $l_2$  do not overlap, as  $\mathcal{R}$  is orthogonal.

1. Let  $l_1 \rightarrow r_1 = l_2 \rightarrow r_2 = l \rightarrow r$ . For any  $\langle x_i, y_j \rangle \in V' \cap W'$ , we have  $\hat{G}^*(\langle x_i, y_j \rangle) = \hat{G}^*(l \rho_{i,j}) = \hat{G}^*(l \delta_{i,j})$ , and, hence,  $\rho_{i,j} = \delta_{i,j}$ . Let

$$\begin{aligned} G'' &= \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin V' \cup W'\} \\ &\quad \cup \{\langle x_i, y_j \rangle = r \rho_{i,j} \mid \langle x_i, y_j \rangle \in V' \cap W'\} \\ &\quad (\text{which is equivalent to } \{\langle x_i, y_j \rangle = r \delta_{i,j} \mid \langle x_i, y_j \rangle \in V' \cap W'\}) \\ &\quad \cup \{\langle x_i, y_j \rangle = r \rho_{i,j} \mid \langle x_i, y_j \rangle \in V' \setminus W'\} \\ &\quad \cup \{\langle x_i, y_j \rangle = r \delta_{i,j} \mid \langle x_i, y_j \rangle \in W' \setminus V'\}. \end{aligned}$$

Then,  $t_1 = G_1'^*(\langle x, y \rangle) \rightarrow_{l \rightarrow r}^{\Delta'} G''^*(\langle x, y \rangle)$ , where  $\Delta' = \mathcal{SP}_{G_1'}(W' \setminus V')$ , and  $t_2 = G_2'^*(\langle x, y \rangle) \rightarrow_{l \rightarrow r}^{\Gamma'} G''^*(\langle x, y \rangle)$ , where  $\Gamma' = \mathcal{SP}_{G_2'}(V' \setminus W')$ . Furthermore,  $\Delta'$ ,  $\Gamma'$ , and  $G''$  can be constructed effectively.

2. Let  $l_1 \rightarrow r_1 \neq l_2 \rightarrow r_2$ . Since  $\hat{G}^*(\langle x_i, y_j \rangle) = \hat{G}^*(l_1 \rho_{i,j})$  for any  $\langle x_i, y_j \rangle \in V'$  and  $\hat{G}^*(\langle x_i, y_j \rangle) = \hat{G}^*(l_2 \delta_{i,j})$  for any  $\langle x_i, y_j \rangle \in W'$  and since  $l_1$  and  $l_2$  are not variables, we obtain  $V' \cap W' = \emptyset$ . Let

$$\begin{aligned} G'' &= \{\langle x_i, y_j \rangle = w \in \hat{G} \mid \langle x_i, y_j \rangle \notin V' \cup W'\} \\ &\quad \cup \{\langle x_i, y_j \rangle = r_1 \rho_{i,j} \mid \langle x_i, y_j \rangle \in V'\} \\ &\quad \cup \{\langle x_i, y_j \rangle = r_2 \delta_{i,j} \mid \langle x_i, y_j \rangle \in W'\}. \end{aligned}$$

Then,  $t_1 = G_1'^*(\langle x, y \rangle) \rightarrow_{l_2 \rightarrow r_2}^{\Delta'} G''^*(\langle x, y \rangle)$ , where  $\Delta' = \mathcal{SP}_{G_1'}(W')$ , and  $t_2 = G_2'^*(\langle x, y \rangle) \rightarrow_{l_1 \rightarrow r_1}^{\Gamma'} G''^*(\langle x, y \rangle)$ , where  $\Gamma' = \mathcal{SP}_{G_2'}(V')$ . Furthermore,  $\Delta'$ ,  $\Gamma'$ , and  $G''_{\langle x, y \rangle}$  can be constructed effectively.  $\square$

*Example 5.7.* Let  $E = \{x = f(y), y = f(x)\}$ ,  $F = \{x' = f(y'), y' = f(z'), z' = f(x')\}$ ,  $s = E^*(x) = F^*(x')$ ,  $\Delta = \{1 \cdot 1^{2n} \mid n \geq 0\}$ ,  $\Gamma = \{11 \cdot 1^{3n} \mid n \geq 0\}$ , and  $\mathcal{R} = \{f(x) \rightarrow g(x)\}$ . We have  $s \rightarrow^\Delta t_1 = \{x = f(y), y = g(x)\}^*(x)$  with  $\Delta = \mathcal{SP}_{E_x}(\{y\})$  and  $s \rightarrow^\Gamma t_2 = \{x' = f(y'), y' = f(z'), z' = g(x')\}^*(x')$  with  $\Gamma = \mathcal{SP}_{F_{x'}}(\{z'\})$ . The term  $u$  with  $t_1 \rightarrow u \leftarrow t_2$  is now constructed as follows: First we take product of  $E$  and  $F$ . We get

$$\begin{aligned} (E \times F) \upharpoonright \langle x, x' \rangle &= \\ &\quad \{\langle x, x' \rangle = f(\langle y, y' \rangle), \langle y, y' \rangle = f(\langle x, z' \rangle), \langle x, z' \rangle = f(\langle y, x' \rangle), \\ &\quad \langle y, x' \rangle = f(\langle x, y' \rangle), \langle x, y' \rangle = f(\langle y, z' \rangle), \langle y, z' \rangle = f(\langle x, x' \rangle)\}. \end{aligned}$$

We now have  $\Delta = (E \times F)_{\langle x, x' \rangle}(V)$  with  $V = \{\langle y, x' \rangle, \langle y, y' \rangle, \langle y, z' \rangle\}$  and  $\Gamma = (E \times F)_{\langle x, x' \rangle}(W)$  with  $W = \{\langle x, z' \rangle, \langle y, z' \rangle\}$ . Hence, we define

$$u = \{\langle x, x' \rangle = f(\langle y, y' \rangle), \langle y, y' \rangle = g(\langle x, z' \rangle), \langle x, z' \rangle = g(\langle y, x' \rangle), \\ \langle y, x' \rangle = g(\langle x, y' \rangle), \langle x, y' \rangle = f(\langle y, z' \rangle), \langle y, z' \rangle = g(\langle x, x' \rangle)\}^*(\langle x, x' \rangle)$$

and we obtain  $t_1 \xrightarrow{\Delta'} u$  with  $\Delta' = (E \times F)_{\langle x, x' \rangle}(W \setminus V)$  and  $t_2 \xrightarrow{\Gamma'} u$  with  $\Gamma' = (E \times F)_{\langle x, x' \rangle}(V \setminus W)$ .

Lemma 5.4 allows us to define rewrite sequences  $s_1 \xrightarrow{*}_{\mathcal{R}} s_n$  which are given by  $\langle s_i, \Delta_i, l_i \rightarrow r_i \rangle_{1 \leq i \leq n-1}$  with  $s_i \xrightarrow{\Delta_i}_{l_i \rightarrow r_i} s_{i+1}$  for each  $1 \leq i \leq n-1$ .

**Theorem 5.8 (Confluence).** *Let  $\mathcal{R}$  is an orthogonal TRS. Given rewrite sequences  $s \xrightarrow{*}_{\mathcal{R}} t_1$  and  $s \xrightarrow{*}_{\mathcal{R}} t_2$ , a rational term  $u$  and rewrite sequences  $t_1 \xrightarrow{*}_{\mathcal{R}} u$  and  $t_2 \xrightarrow{*}_{\mathcal{R}} u$  can be constructed effectively.*

*Proof.* Immediate by Lemma 5.6. □

The following example shows that weakly orthogonal TRS may be non-confluent in rational term rewriting. Hence, the above theorem cannot be extended to weakly orthogonal rewriting.

*Example 5.9.* Let  $\mathcal{R} = \{\mathbf{p}(s(x)) \rightarrow x, \mathbf{s}(p(x)) \rightarrow x\}$ . There are two critical pairs in  $\mathcal{R}$ ,  $\langle \mathbf{p}(x), \mathbf{p}(x) \rangle$  and  $\langle \mathbf{s}(x), \mathbf{s}(x) \rangle$ , which are both trivial. Hence,  $\mathcal{R}$  is weakly orthogonal. Suppose  $t = \{x = \mathbf{p}(s(x))\}^*(x)$ . We have  $t \rightarrow_{\mathcal{R}} \perp$  and  $t \rightarrow_{\mathcal{R}} \mathbf{p}(\perp)$ . Since  $\perp$  and  $\mathbf{p}(\perp)$  are distinct normal forms, they are not joinable. Hence,  $\mathcal{R}$  is not confluent.

Non-confluence of weakly orthogonal systems may seem remarkable from the perspective of finitary rewriting, especially since confluence for weakly orthogonal systems does hold in a term graph rewriting formalism that allows for copying of subterms [1]. However, it is less remarkable from the infinitary perspective: A similar counterexample can be constructed within the approach to infinitary rewriting as taken by Kennaway et al.; that counterexample can be found in [8].

## 6 Conclusion

We have introduced rational term rewriting based on regular systems. Unlike [5,6], which is specialized to orthogonal rewrite systems, our definition is amenable to arbitrary rewrite systems. We have shown that our notion of rewrite steps is characterized by rational patterns. Moreover, we have shown decidable properties concerning our rewrite steps and constructive confluence for orthogonal rewrite systems, i.e., that for two rewrite sequences  $s \xrightarrow{*} t_1$  and  $s \xrightarrow{*} t_2$  we can effectively construct a regular system representing a rational term  $u$  and rewrite sequences  $t_1 \xrightarrow{*} u$  and  $t_2 \xrightarrow{*} u$ . We have also given an example showing that rational term rewriting is in general not confluent for weakly orthogonal rewrite systems.

It is important to know for which classes of TRSs  $\mathcal{R}$ ,  $s \xleftrightarrow{*}_{\mathcal{R}} t$  can be decided. Even for an orthogonal TRS  $\mathcal{R}$ , our results do not immediately imply decidability of  $s \xleftrightarrow{*}_{\mathcal{R}} t$ . As future work, we plan to address this problem.

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## References

1. Z. M. Ariola and J. W. Klop. Equational term graph rewriting. *Fundamenta Informaticae*, 26:207–240, 1996.
2. F. Baader and T. Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.
3. E. Barendsen. Term graph rewriting. In Terese [17], Chapter 13, pages 712–743.
4. A. Corradini. Term rewriting in  $CT_{\Sigma}$ . In *CAAP 1993*, volume 668 of *LNCS*, pages 468–484, 1993.
5. A. Corradini and F. Drewes. Term graph rewriting and parallel term rewriting. In *TERMGRAPH 2011*, volume 48 of *EPTCS*, pages 3–18, 2011.
6. A. Corradini and F. Gadducci. Rational term rewriting. In *FoSSaCS 1998*, volume 1378 of *LNCS*, pages 156–171, 1998.
7. B. Courcelle. Fundamental properties of infinite trees. *Theoretical Computer Science*, 25(2):95–169, 1983.
8. J. Endrullis, C. Grabmayer, D. Hendriks, J. W. Klop, and V. van Oostrom. Unique normal forms in infinitary weakly orthogonal rewriting. In *RTA 2010*, volume 6 of *LIPICs*, pages 85–102. Schloss Dagstuhl, 2010.
9. J. A. Goguen, J. W. Thatcher, E. G. Wagner, and J. B. Wright. Initial algebra semantics and continuous algebras. *Journal of the ACM*, 24(1):68–95, 1977.
10. P. Inverardi and M. V. Zilli. Rational rewriting. In *MFCS 1994*, volume 841 of *LNCS*, pages 433–442, 1994.
11. J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries. On the adequacy of graph rewriting for simulating term rewriting. *ACM Transactions on Programming Languages and Systems*, 16(3):493–523, 1994.
12. R. Kennaway and F.-J. de Vries. Infinitary rewriting. In Terese [17], Chapter 12, pages 668–711.
13. R. Kennaway, J. W. Klop, R. Sleep, and F.-J. de Vries. Transfinite reductions in orthogonal term rewriting systems. *Information and Computation*, 119(1):18–38, 1995.
14. J. Ketema and J. G. Simonsen. Computing with infinite terms and infinite reductions. Unpublished manuscript.
15. D. Plump. Collapsed tree rewriting: Completeness, confluence, and modularity. In *CTRS 1993*, volume 656 of *LNCS*, pages 97–111, 1993.
16. D. Plump. Term graph rewriting. In *Handbook of Graph Grammars and Computing by Graph Transformation Volume 2: Applications, Languages and Tools*, pages 3–61. World Scientific, 1999.
17. Terese, editor. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.
18. K. Weihrauch. *Computable Analysis: An Introduction*. Springer-Verlag, 2000.